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Total No. of Pages : 02

Total No. of Questions : 09

B.Tech. (AI & DS/ AI & ML/ Block Chain / CSE / CE / CS / Computer Science and Design / EEE / EE / ECE / Electronics & Telecommunication Engineering / FT / IT / ME / Robotics & Artificial Intelligence/ Internet of Things and Cyber Security including Block Chain Technology) (Sem.- 1)

### ENGINEERING MATHEMATICS-I

Subject Code : BTAM101-23

M.Code : 93796

Date of Examination: 08-12-2023

Time : 3 Hrs.

Max. Marks : 60

#### INSTRUCTIONS TO CANDIDATES :

1. SECTION-A is COMPULSORY consisting of TEN questions carrying TWO marks each.
2. SECTION - B & C have FOUR questions each.
3. Attempt any FIVE questions from SECTION B & C carrying EIGHT marks each.
4. Select atleast TWO questions from SECTION - B & C.

#### SECTION-A

1. Answer briefly :

a) What do you mean by bounded and unbounded sequences?

b) Prove that the sequence  $\frac{2n-7}{3n+2}$  is monotonically increasing.

c) Define  $p$ -Test for the series.

d) Find the length of the arc of the parabola  $x^2 = 4ay$  extending from the vertex to one extremity of the latus rectum.

e) Test for convergence of integral  $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$ .

f) Define Beta function.

g) State Euler's theorem for homogeneous function.

h) Show that the function  $f(x, y) = \frac{2x^2y}{x^4 + y^2}$  has no limit as  $(x, y) \rightarrow (0, 0)$ .

i) Evaluate  $\int_1^{\log 8} \int_0^{\log y} e^{x+y} dx dy$ .

j) Evaluate  $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz dz dy dx$ .

### SECTION-B

2. Prove that the sequence  $a_n$  where  $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$  is convergent.
3. Discuss the convergence or divergence of the series  $\frac{1}{1.2.3} + \frac{2}{2.3.4} + \frac{5}{3.4.5} + \dots$
4. Find the surface of the solid generated by the revolution of the ellipse  $x^2 + 4y^2 = 16$  about its major axis.
5. Prove that  $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$  where  $m > 0, n > 0$ .

### SECTION-C

6. If  $V = r^m$  where  $r^2 = x^2 + y^2 + z^2$ , show that  $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$ .
7. Show that the first four terms of the Maclaurin's expansion of  $e^{ax} \cos by$  are :

$$1 + ax + \frac{1}{2}(a^2x^2 - b^2y^2) + \frac{1}{6}(a^3x^3 - 3ab^2xy^2)$$

8. Evaluate  $\int_0^{\infty} \int_0^x xe^{-x^2/y} dy dx$  by change of order of integration.

9. Evaluate  $\iiint \frac{dx dy dz}{(x+y+z+1)^3}$  over the region  $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$

**NOTE :** Disclosure of Identity by writing Mobile No. or Marking of passing request on any paper of Answer Sheet will lead to UMC against the Student.

B.Tech (Semester-1) (Dec, 2023)

ENGG. MATHEMATICS-I

Subject Code: BTAM-101-23

M. Code: 93796

B.Tech (AI&DS/AI&ML/CSE/EE/ECE/CSE-DS/IT)

## SECTION-A

Q1 Answer briefly:-

(a) What do you mean by bounded and unbounded sequence?  
→ A sequence  $\{a_n\}$  is said to be bounded above if there exist a real no. ' $k$ ' such that  $a_n \leq k \forall n \in \mathbb{N}$ ,  $k$  is called an upper bound of  $\{a_n\}$ .

A sequence  $\{a_n\}$  is said to be bounded below if there exist a real no. ' $h$ ' such that  $h \leq a_n \forall n \in \mathbb{N}$ ,  $h$  is called an lower bound of  $\{a_n\}$ .

A sequence  $\{a_n\}$  is said to be bounded if it is bounded above as well as bounded below i.e. if  $\exists$  two real no. ' $h$ ' and ' $k$ ' such that:-

$$h \leq a_n \leq k, n \in \mathbb{N}$$

A sequence is said to be unbounded if it is not bounded. A sequence is unbounded if given  $\Delta > 0$  however large  $\exists m \in \mathbb{N}$  such that  $|a_n| > \Delta \forall n \geq m$ .

(b) Prove that the sequence  $\frac{2n-7}{3n+2}$  is monotonically increasing.

→ A sequence  $\{a_n\}$  is said to be monotonically increasing if  $a_n \leq a_{n+1}$ .

Here,  $a_n = \frac{2n-7}{3n+2}$

$$a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2}$$

$$= \frac{2n+2-7}{3n+3+2}$$

$$= \frac{2n-5}{3n+5}$$

$$a_{n+1} - a_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$$

$$= \frac{(2n-5)(3n+2) - (2n-7)(3n+5)}{(3n+5)(3n+2)}$$

$$= \frac{6n^2 + 4n - 15n - 10 - [6n^2 + 10n - 21n - 35]}{(3n+5)(3n+2)}$$

$$= \frac{6n^2 - 11n - 10 - 6n^2 + 11n + 35}{(3n+5)(3n+2)}$$

$$= \frac{25}{(3n+5)(3n+2)} > 0$$

$$= a_{n+1} - a_n > 0$$

$$= a_{n+1} > a_n$$

$\therefore \{a_n\}$  is monotonically increasing sequence.

Hence Proved

(c) Define p-Test for the series:-

$$\sum_{n=1}^{\infty} \frac{1}{n^p} > 0, \text{ cgs if } p > 1$$

dgs if  $p < 1$

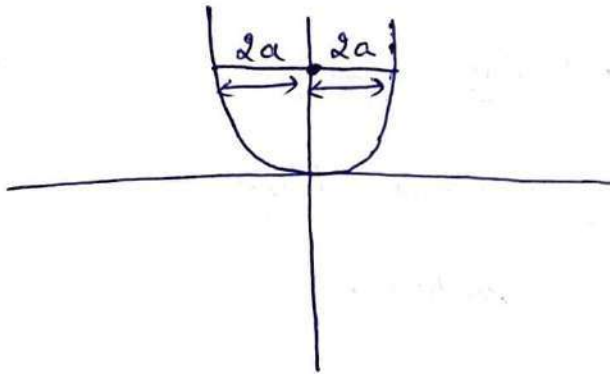
for eg:-  $\sum \frac{1}{n^2}$ , here  $p=2 > 1$

$\therefore \sum a_n$  cgs

$$\sum \frac{1}{n^{1/2}} = \text{Here, } p = \frac{1}{2} < 1$$

$\therefore$  S an dgs

(d) Find the length of the arc of the parabola  $x^2 = 4ay$  extending from the vertex to one extremity of the latus rectum.



Given,  $x^2 = 4ay$

$$\rightarrow y = \frac{x^2}{4a}$$

$$\rightarrow \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

$$\begin{aligned} \text{Length of the arc} &= \int_0^{2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^{2a} \sqrt{1 + \left(\frac{x}{2a}\right)^2} dx \\ &= \frac{1}{2a} \int_0^{2a} \sqrt{4a^2 + x^2} dx \end{aligned}$$

By using formula,  $\int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log |x + \sqrt{a^2 + x^2}|$

$$= \frac{1}{2a} \left[ \frac{x}{2} \sqrt{(2a)^2 + x^2} + \frac{(2a)^2}{2} \log |x + \sqrt{(2a)^2 + x^2}| \right]_0^{2a}$$

$$= \frac{1}{2a} \left[ \frac{2a}{2} \sqrt{(2a)^2 + (2a)^2} + \frac{4a^2}{2} \log |2a + \sqrt{(2a)^2 + (2a)^2}| \right] - \frac{1}{2a} \left[ 0 + \frac{(2a)^2}{2} \log |0 + \sqrt{(2a)^2 + (0)^2}| \right]$$

$$= \frac{1}{2a} \left[ a \sqrt{4a^2 + 4a^2} + 2a^2 \log |2a + \sqrt{4a^2 + 4a^2}| \right] - \frac{1}{2a} \left[ \frac{4a^2}{2} \log |2a| \right]$$

$$= \frac{1}{2a} \left[ a \sqrt{8a^2} + 2a^2 \log |2a + \sqrt{8a^2}| \right] - \frac{1}{2a} \left[ 2a^2 \log 2a \right]$$

$$= \frac{1}{2a} \left[ 2\sqrt{2}a^2 + 2a^2 \log |2a + 2\sqrt{2}a| \right] - \frac{1}{2a} \left[ 2a^2 \log 2a \right]$$

$$= a \left[ \sqrt{2} + \log \frac{|2a + 2\sqrt{2}a|}{2a} \right]$$

$$= a \left[ \sqrt{2} + \log (1 + \sqrt{2}) \right] \rightarrow \underline{\underline{\text{Ans}}}$$

(e) Test for convergence of integral  $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$

$\rightarrow \sin^2 x$  will always be positive.

By p-test:  $\rightarrow 2 > 1$

$\therefore$  Integral  $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$  converges.

(f) Define Beta function.

The integral  $I = \int_0^1 x^{m-1} (1-x)^{n-1} dx$  where  $m > 0$  and  $n > 0$  is called a beta function and is denoted by  $\beta(m, n)$ .

$$\text{Eg: } \int_0^1 x^3 (1-x)^5 dx$$

$$\text{Here, } m-1 = 3$$

$$\boxed{m=4}$$

$$n-1 = 5$$

$$\boxed{n=6}$$

$$\beta(m, n) = \beta(4, 6)$$

$$\text{Eg 2: } \int_0^1 x^{1/3} (1-x)^{-2} dx$$

$$1/2a^m, m-1 = \frac{1}{3}$$

$$m = \frac{1}{3} + 1$$

$$\boxed{m = \frac{4}{3}}$$

$$n-1 = -2$$

$$n = -2 + 1$$

$$\boxed{n = -1}$$

As  $n < 0$ , due to which condition of Beta function is not satisfied.  $\therefore$  It is not a beta function.

(g) State Euler's theorem for homogeneous function.

If  $z$  is homogeneous function of  $x$  and  $y$  of order  $n$

then  $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \quad \forall x, y \in \text{Domain of the function.}$

For eg:-  $z = \frac{x^3 y^3}{x^3 + y^3}$

$$\rightarrow z = \frac{x^3 \left(\frac{y}{x}\right)^3 x^3}{x^3 \left(1 + \left(\frac{y}{x}\right)^3\right)}$$

Here,  $z$  is a homogeneous function of degree 3.

By using Euler's theorem:-

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3z,$$

where  $\boxed{n=3}$

(h) Show that the function  $f(x, y) = \frac{2x^2 y}{x^4 + y^2}$  has no limit as  $(x, y) \rightarrow (0, 0)$ .

$$\rightarrow \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(h)^2 \times 0 - 0}{(h)^4 + (0)^2}$$

$$= \lim_{h \rightarrow 0} \frac{0}{h}$$

$$= 0$$

$$\left(\frac{\partial f}{\partial y}\right)_{0,0} = \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{f(0, 0+k) - f(0, 0)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{2(0)^2 \times k - 0}{(0)^4 + k^2}$$

$$= \lim_{k \rightarrow 0} \frac{0}{k}$$

$$= 0$$

Hence,  $f(x, y) = \frac{2x^2y}{x^4 + y^2}$  has no limit as  $(x, y) \rightarrow (0, 0)$ .

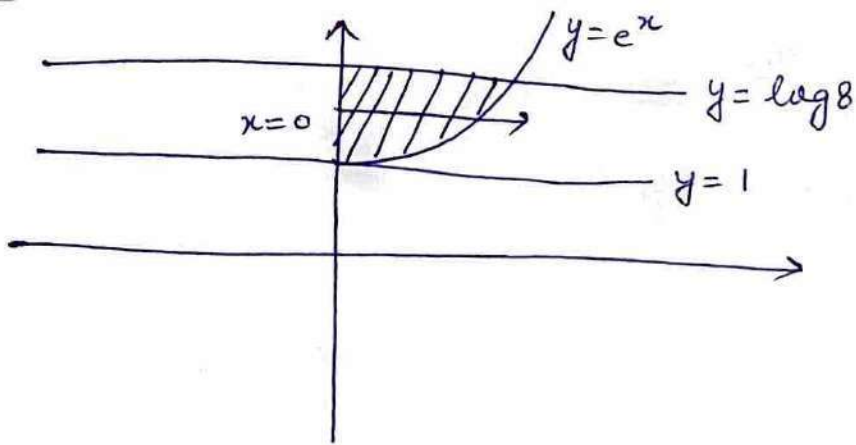
(i) Evaluate  $\int_1^{\log_e 8} \int_0^{\log_e y} e^{x+y} dx dy$

→ Region of integration :-

$$x=0 \text{ to } x=\log_e y$$

$$y=1 \text{ to } y=\log_e 8$$





$$\rightarrow \int_1^{\log 8} \left[ \int_0^{\log y} e^{x+y} dx \right] dy$$

$$\rightarrow \int_1^{\log 8} \left[ e^y [e^x]_0^{\log y} \right] dy$$

$$\rightarrow \int_1^{\log 8} e^y [e^{\log y} - e^0] dy$$

$$\rightarrow \int_1^{\log 8} e^y [y - 1] dy$$

I                  II

By using ILATE

$$\rightarrow [(y-1)e^y]_1^{\log 8} - \int_1^{\log 8} 1 \cdot e^y dy$$

$$\rightarrow [(y-1)e^y]_1^{\log 8} - [e^y]_1^{\log 8}$$

$$\rightarrow (\log 8 - 1)e^{\log 8} - [e^{\log 8} - e^1]$$

$$\rightarrow (\log 8 - 1)e^{\log 8} - [e^{\log 8} - e]$$

$$\rightarrow (\log 8 - 1)e^{\log 8} - e^{\log 8} + e$$

$$\rightarrow e^{\log 8} [\log 8 - 1 - 1] + e$$

$$\rightarrow e^{\log 8} [\log 8 - 2] + e$$

$$\rightarrow 8 [-\log 8 - 2] + e$$

$$\rightarrow 8 \log 8 - 16 + e \quad \underline{\text{Ans}}$$

(j) Evaluate :-  $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz \, dz \, dy \, dx$

$$\rightarrow \int_0^1 \int_0^{1-x} \left[ \frac{xyz^2}{2} \right]_0^{2-x} dy \, dx$$

$$\rightarrow \int_0^1 \int_0^{1-x} \left[ \frac{xy(2-x)^2}{2} dy \right] dx$$

$$\rightarrow \int_0^1 \left[ \frac{x(2-x)^2}{2} \cdot \frac{y^2}{2} \right]_0^{1-x} dx$$

$$\rightarrow \int_0^1 \frac{x(2-x)^2 (1-x)^2}{2} dx$$

$$\rightarrow \frac{1}{4} \int_0^1 x(2-x)^2 (1-x)^2 dx$$

$$\rightarrow \frac{1}{4} \int_0^1 x(4+x^2-4x)(1+x^2-2x) dx$$

$$\rightarrow \frac{1}{4} \int_0^1 (4x+x^3-4x^2)(1+x^2-2x) dx$$

$$\rightarrow \frac{1}{4} \int_0^1 4x+4x^3-8x^2+x^3+x^5-2x^4-4x^2-4x^4+8x^3 dx$$

$$\rightarrow \frac{1}{4} \int_0^1 4x+13x^3-12x^2-6x^4+x^5 dx$$

$$\rightarrow \frac{1}{4} \left[ \frac{4x^2}{2} + \frac{13x^4}{4} - \frac{12x^3}{3} - \frac{6x^5}{5} + \frac{x^6}{6} \right]_0^1$$

$$\rightarrow \frac{1}{4} \left[ 2 + \frac{13}{4} - \frac{12}{3} - \frac{6}{5} + \frac{1}{6} \right]$$

$$\rightarrow \frac{1}{4} \left[ -2 + \frac{13}{4} - \frac{6}{5} + \frac{1}{6} \right]$$

$$\rightarrow \frac{1}{4} \left[ -2 + \frac{65-24}{20} + \frac{1}{6} \right]$$

$$\rightarrow \frac{1}{4} \left[ -2 + \frac{41}{20} + \frac{1}{6} \right]$$

$$\rightarrow \frac{1}{4} \left[ \frac{-120 + 123 + 10}{60} \right]$$

$$\rightarrow \frac{1}{4} \left[ \frac{13}{60} \right]$$

$$\rightarrow \frac{13}{240} \quad \underline{\text{Ans}}$$

### Section-B

Q2 Prove that the sequence where  $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$  is convergent.

$$\rightarrow a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$$

$$\rightarrow a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$\rightarrow a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} - \left[ \frac{1}{2(n+1)} - \frac{1}{(n+1)} \right]$$

$$= \frac{1}{2n+1} - \frac{1}{2(n+1)}$$

$$= \frac{2n+2 - 2n-1}{(2n+1)(2n+2)}$$

$$= \frac{1}{(2n+1)(2n+2)} > 0$$

Since,  $a_{n+1} > a_n$

$\therefore$  The given sequence is monotonically increasing

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} < \frac{1}{n} + \frac{1}{n} + \dots \text{ n terms}$$

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} < \frac{1}{n} (\times)$$

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} < 1$$

For  $\{a_n\}$  upper bound exists

Hence, sequence  $\{a_n\}$  is bounded above

So, monotonically increasing + bounded above

$\therefore$  Convergent

Hence Proved

Q3 Discuss the convergence or divergence of the series

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{2 \cdot 3 \cdot 4} + \dots$$

$\rightarrow$  Here,  $a_n = \frac{2n-1}{n(n+1)(n+2)}$

Let  $b_n = \frac{1}{n^2}$

So,  $\frac{a_n}{b_n} = \frac{(2n-1)}{n(n+1)(n+2)} \times n^2$

$$= \frac{a_n}{b_n} = \frac{n^3 \left(2 - \frac{1}{n}\right)}{n^3 \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}$$

$$= \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 2$$

which is finite and non-zero.

∴ By comparison test both  $\sum a_n$  and  $\sum b_n$  converge or diverge together.

But by p-test  $\sum b_n$  cgs as  $2 > 1$

∴ By comparison test

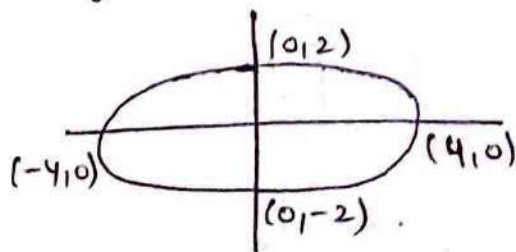
$\sum a_n$  converges

Q4 Find the surface of the solid generated by the revolution of the ellipse  $x^2 + 4y^2 = 16$  about its major axis.

→  $x^2 + 4y^2 = 16$  (Given)

→  $\frac{x^2}{16} + \frac{4y^2}{16} = 1$

→  $y^2 = \frac{16 - x^2}{4}$  or  $4x^2 + 16y^2 = 64$



Thus  $16y^2 = 64 - 4x^2$

As  $x^2 + 4y^2 = 16$

→  $2x + 8y \frac{dy}{dx} = 0$

∴  $\left[ \frac{dy}{dx} = -\frac{x}{4y} \right]$

Thus,  $\left( \frac{dy}{dx} \right)^2 = \frac{x^2}{16y^2}$

Now,  $1 + \left( \frac{dy}{dx} \right)^2 = \frac{16y^2 + x^2}{16y^2}$

$\frac{ds}{dx} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2}$

$\frac{ds}{dx} = \sqrt{\frac{16y^2 + x^2}{16y^2}}$

$\frac{ds}{dx} = \frac{\sqrt{16y^2 + x^2}}{\sqrt{(4)^2 y^2}}$

$$\frac{ds}{dx} = \frac{\sqrt{64-4x^2+x^2}}{64-4x^2}$$

$$\frac{ds}{dx} = \frac{\sqrt{64-3x^2}}{\sqrt{16y^2}}$$

$$\frac{ds}{dx} = \frac{\sqrt{64-3x^2}}{4y} \quad - \textcircled{1}$$

$$S = 2\pi \int_{-4}^4 y ds = 2\pi \int_{-4}^4 y \frac{\sqrt{64-3x^2}}{4y} dx$$

$$S = \frac{2\pi}{4} \int_{-4}^4 \sqrt{64-3x^2} dx$$

$$S = \frac{2\pi}{4} \cdot 2 \cdot \int_0^4 \sqrt{64-3x^2} dx$$

$$S = \pi \cdot \sqrt{3} \int_0^4 \sqrt{\frac{64}{3}-x^2} dx$$

$$S = \pi \cdot \sqrt{3} \left[ \frac{x \sqrt{\frac{64}{3}-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{\sqrt{64/3}} \right]$$

$$S = \pi \cdot \sqrt{3} \left[ \frac{4 \sqrt{\frac{64}{3}-16}}{2} + \frac{64}{3 \cdot 2} \sin^{-1} \frac{4\sqrt{3}}{6} - 0 \right]$$

$$S = \sqrt{3} \pi \left[ 2 \sqrt{\frac{16}{3}} + \frac{32 \cdot \pi}{3 \cdot 3} \right]$$

$$S = \sqrt{3} \pi \left[ \frac{2 \times 4}{\sqrt{3}} + \frac{32\pi}{9} \right]$$

$$S = \sqrt{3} \pi \left[ \frac{8}{\sqrt{3}} + \frac{32\pi}{9} \right]$$

$$S = \left[ 8\pi + \frac{32\sqrt{3}\pi^2}{9} \right] \rightarrow \underline{\underline{\text{Answer}}}$$

Prove that  $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ , where  $m > 0, n > 0$ .

→ Proof :-  $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

Put  $x = tz$

$dx = t dz$

When  $x = 0, z = 0$

$x = \infty, z = \infty$

$$\begin{aligned} \Gamma(n) &= \int_0^{\infty} (tz)^{n-1} e^{-tz} t dz \\ &= \int_0^{\infty} t^n z^{n-1} e^{-tz} dz \end{aligned}$$

Multiply both sides by  $e^{-t} t^{m-1}$

$$\rightarrow \Gamma(n) e^{-t} t^{m-1} = e^{-t} t^{m-1} \int_0^{\infty} t^n z^{n-1} e^{-tz} dz$$

$$\rightarrow \Gamma(n) e^{-t} t^{m-1} = \int_0^{\infty} t^{m+n-1} z^{n-1} e^{-t(1+z)} dz$$

Integrating both sides from 0 to  $\infty$  w.r.t. 't' on both sides :-

$$\rightarrow \Gamma(n) \int_0^{\infty} e^{-t} t^{m-1} dt = \int_0^{\infty} z^{n-1} \left[ \int_0^{\infty} t^{m+n-1} e^{-t(z+1)} dt \right] dz$$

$$\rightarrow \Gamma(n) \Gamma(m) = \int_0^{\infty} z^{n-1} \left[ \int_0^{\infty} t^{m+n-1} e^{-t(z+1)} dt \right] dz$$

Put  $t(z+1) = y$

$$\rightarrow t = \frac{y}{z+1}$$

$$dt = \frac{dy}{z+1}$$

$$\text{when } t=0 \quad | \quad y=0$$

$$t=\infty \quad | \quad y=\infty$$

$$\rightarrow \Gamma(n) \Gamma(m) = \int_0^{\infty} z^{n-1} \left[ \int_0^{\infty} \left( \frac{y}{z+1} \right)^{m+n-1} e^{-y} \cdot \frac{dy}{z+1} \right] dz$$

$$\rightarrow \Gamma(n) \Gamma(m) = \int_0^{\infty} \frac{z^{n-1}}{(z+1)^{m+n}} \left[ \int_0^{\infty} y^{m+n-1} e^{-y} dy \right] dz$$

$$\rightarrow \Gamma(n) \Gamma(m) = \Gamma(m+n) \int_0^{\infty} \frac{z^{n-1}}{(1+z)^{m+n}} dz$$

$$\rightarrow \frac{\Gamma(n) \Gamma(m)}{\Gamma(m+n)} = \beta(m, n) \quad \left[ \text{By property 2 of Beta function} \right]$$

### Section - c

Q6 If  $v = r^m$  where  $r^2 = x^2 + y^2 + z^2$ , show that  $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m(m+1)r^{m-2}$ .

$$\rightarrow r = (x^2 + y^2 + z^2)^{1/2}$$

$$\rightarrow r^m = (x^2 + y^2 + z^2)^{m/2}$$

$$\rightarrow v = (x^2 + y^2 + z^2)^{m/2}$$

$$\rightarrow \frac{\partial v}{\partial x} = \frac{m}{2} (x^2 + y^2 + z^2)^{\frac{m-2}{2}} \cdot x \cdot 2x$$

$$\rightarrow \frac{\partial v}{\partial x} = mx (x^2 + y^2 + z^2)^{\frac{m-2}{2}}$$

$$\rightarrow \frac{\partial^2 v}{\partial x^2} = m \left[ x \cdot \frac{m-2}{2} (x^2 + y^2 + z^2)^{\frac{m-4}{2}} \cdot x \cdot 2x \right] + \left[ (x^2 + y^2 + z^2)^{\frac{m-2}{2}} \cdot 1 \right]$$

$$\rightarrow \frac{\partial^2 v}{\partial x^2} = m \left[ x^2 (m-2) (x^2 + y^2 + z^2)^{\frac{m-4}{2}} \right] + \left[ (x^2 + y^2 + z^2)^{\frac{m-2}{2}} \right]$$



$$= m(m-2)x^2(x^2+y^2+z^2)^{\frac{m-4}{2}} + m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$\frac{\partial^2 V}{\partial y^2} = m(m-2)y^2(x^2+y^2+z^2)^{\frac{m-4}{2}} + m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$\frac{\partial^2 V}{\partial z^2} = m(m-2)z^2(x^2+y^2+z^2)^{\frac{m-4}{2}} + m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m-2)(x^2+y^2+z^2)^{\frac{m-4}{2}}(x^2+y^2+z^2) + 3m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$= m(m-2)x^{m-4}x^2 + 3m(x)^{m-2}$$

$$= m(m-2)x^{m-2} + 3mx^{m-2}$$

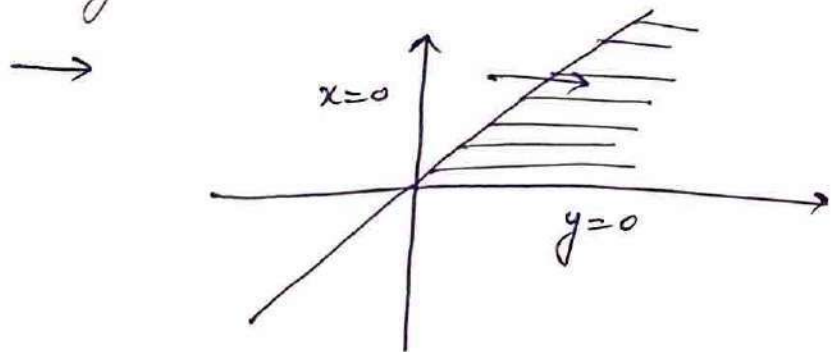
$$= [m(m-2) + 3m]x^{m-2}$$

$$= m[m-2+3]x^{m-2}$$

$$= m[m+1]x^{m-2} = \text{R.H.S}$$

Hence verified

Q8 Evaluate  $\int_0^{\infty} \int_0^x xe^{-x^2/y} dy dx$  by change of order of integration.



$$y=0, y=x \text{ \& } x=0, x=\infty$$

changing the order:-

$$\rightarrow \int_0^{\infty} \int_y^{\infty} xe^{-x^2/y} dx dy$$

$$\text{Put } x^2 = t$$

$$\rightarrow 2ndn = dt$$

$$\rightarrow xdx = \frac{dt}{2}$$

$$\rightarrow \frac{1}{2} \int_0^{\infty} \left[ \int_{y^2}^{\infty} e^{-t/y} dt \right] dy$$

$$\rightarrow \frac{1}{2} \int_0^{\infty} \left[ \frac{e^{-t/y}}{-1/y} \right]_{y^2}^{\infty} dy$$

$$\rightarrow \frac{1}{2} \int_0^{\infty} \left[ -ye^{-t/y} \right]_{y^2}^{\infty} dy$$

$$\rightarrow -\frac{1}{2} \int_0^{\infty} \left[ y \left[ e^{-\infty} - e^{-y^2/y} \right] \right] dy$$

$$\rightarrow -\frac{1}{2} \int_0^{\infty} y \left[ 0 - e^{-y} \right] dy$$

$$\rightarrow \frac{1}{2} \int_0^{\infty} ye^{-y} dy$$

$$\rightarrow \frac{1}{2} \left[ y \frac{e^{-y}}{-1} \right]_0^{\infty} - \int_0^{\infty} 1 \cdot \frac{e^{-y}}{-1} dy$$

$$\rightarrow \frac{1}{2} \left[ -ye^{-y} \right]_0^{\infty} + \int_0^{\infty} e^{-y} dy$$

$$\rightarrow \frac{1}{2} \left[ \left[ -\infty e^{-\infty} - 0 \right] + \left[ \frac{e^{-y}}{-1} \right]_0^{\infty} \right]$$

$$\rightarrow \frac{1}{2} \left[ 0 - 0 \right] - \left[ e^{-\infty} - e^0 \right]$$

$$\rightarrow \frac{1}{2} \left[ 0 - [0 - 1] \right]$$

$$= \frac{1}{2} \underline{\underline{\text{Ans}}}$$

Q7 Show that the first four terms of Maclaurin's series Expansion of  $e^{ax} \cos by$  are:

$$1 + ax + \frac{1}{2}(a^2x^2 - b^2y^2) + \frac{1}{3}(a^3x^3 - 3abxy^2)$$

Soln

The Maclaurin series expansion of  $e^{ax} \cos(by)$  can be written by combining the Maclaurin series for  $e^{ax}$  and  $\cos(by)$ .

$$\text{Let } ax = u \text{ and } by = v$$

So Expansions of  $e^u$  and  $\cos(v)$  are given as:

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \dots$$

$$\cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} + \dots$$

Substituting values of 'u' and 'v' back into above Expansions

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \dots$$

$$\cos by = 1 - \frac{(by)^2}{2!} + \frac{(by)^4}{4!} - \frac{(by)^6}{6!} + \dots$$

Multiplying both the series

$$e^{ax} \cos by = \left(1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \dots\right) \left(1 - \frac{(by)^2}{2!} + \frac{(by)^4}{4!} - \frac{(by)^6}{6!} + \dots\right)$$

$$= 1 + ax + \frac{1}{2!}(a^2x^2 - b^2y^2) + \frac{1}{3!}(a^3x^3 - 3abxy^2) + \dots$$

Hence these are 1st four terms of series

Evaluate  $\iiint \frac{dx dy dz}{(x+y+z+1)^3}$  over the region  $x \geq 0$

$$y \geq 0, z \geq 0, x+y+z \leq 1$$

Since  $x+y+z \leq 1, x \geq 0, y \geq 0, z \geq 0$

$$\therefore x \leq 1, x+y \leq 1, x+y+z \leq 1$$

$$\Rightarrow x \leq 1, y \leq 1-x, z \leq 1-x-y$$

$$\therefore V = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y\}$$

$$\therefore \iiint_V \frac{dx dy dz}{(x+y+z+1)^3} = \int_0^1 \int_0^{1-x} \left( \int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3} \right) dy dx$$

$$= \int_0^1 \int_0^{1-x} \left( \int_0^{1-x-y} (x+y+z+1)^{-3} dz \right) dy dx$$

$$= \int_0^1 \int_0^{1-x} \left[ \frac{(x+y+z+1)^{-2}}{-2} \right]_0^{1-x-y} dy dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[ \frac{1}{4} - \frac{1}{(x+y+1)^2} \right] dy dx$$

$$= -\frac{1}{2} \int_0^1 \left[ \frac{1}{4} y - \frac{(x+y+1)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left[ \left( \frac{1-x}{4} + \frac{1}{2} \right) - \left( 0 + \frac{1}{x+1} \right) \right] dx$$

$$= -\frac{1}{8} \int_0^1 \left( 1-x+2 - \frac{4}{x+1} \right) dx$$

$$= -\frac{1}{8} \int_0^1 \left( -x - \frac{4}{x+1} + 3 \right) dx$$

$$= -\frac{1}{8} \left[ -\frac{x^2}{2} - 4 \log(x+1) + 3x \right]_0^1$$

$$= -\frac{1}{8} \left[ \left( -\frac{1}{2} - 4 \log 2 + 3 \right) - (0 - 4 \log 1 + 0) \right]$$

$$= -\frac{1}{8} \left[ \frac{5}{2} - 4 \log 2 \right]$$

$$= \frac{1}{2} \log 2 - \frac{5}{16}$$

hence

$$\iiint \frac{dx dy dz}{(x+y+z+1)^3} = \frac{1}{2} \log 2 - \frac{5}{16}$$